Interpolation by local space curves

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ABSTRACT

A scheme is proposed for the construction of a visually smooth and fair space curve passing through a prescribed sequence of irregularly spaced and non-coplanar points. A major feature is its locality of definition - in a given region the curve depends only on the nearby points. Also the construction is direct, compact and fast. Illustrative examples are briefly described.

1. INTRODUCTION

This paper outlines a scheme for the construction of a "reasonable" smooth curve passing through a given sequence of irregularly spaced non-coplanar points. The aim is to help the compact specification of a non-planar shape to a machine, which is of great interest in computer-aided design and manufacturing technology.

The construction uses a set of space-curve segments, one between each successive pair of the given points (knots) which are assumed to be distinct. Each segment is joined to its neighbours preserving overall C^2 smoothness; tangent and curvature vectors are continuous at the knots.

Previous work on this subject involves typically generalisation of planar spline curves, whose computation often needs extensive iterative procedures to satisfy non-linear conditions. A recent example, from which earlier literature can be traced, is the paper by D. H. Thomas [1]. By contrast we propose here a direct, fast, small and essentially linear construction. Moreover it has the important property of *locality*. That is, the curve in the vicinity of each knot is fixed by only the positions of closely neighbouring knots. Consequently, altering a knot affects only nearby curve segments. Such locality is very desirable in design work where precise and independent control of detail is needed. A spline-like construction, obeying some requirement of overall minimum average curvature, tends to respond *globally* to perturbations.

We concentrate in fact on a scheme with maximum locality. That is, the curve at knot $\vec{V}_i$ is influenced by only $\vec{V}_{i-1}$ and $\vec{V}_{i+1}$. This is unsophisticated, but has its advantages. It means that at $\vec{V}_i$ the fundamental triad of unit vectors $\vec{t}$, $\vec{n}$, $\vec{b}$ [2] is fixed (unless the three knots are collinear) with t parallel to $\vec{V}_{i+1} - \vec{V}_{i-1}$. Therefore estimating constructions by inspection is fairly straightforward.

The scheme is implemented with parametric polynomial curve segments of standard form. Continuity equations for the curve itself, and for tangent $\vec{t}$ and curvature $\kappa\vec{n}$ at a knot involve three vector coefficients of each of the two segments joining there. Therefore to separate the knots and so maintain locality we work with quintic curves (six coefficients). Cubic curves with C^2 smoothness are always global constructions.

At each knot the continuity equations leave two vector coefficients free to be fixed consistent with locality and with other reasonable requirements such as invariance under complete reversal of ordering of knots, and under translations and rotations of coordinates. Also we wish the shape to be preserved under re-scaling; moreover symmetrically placed knots must give a symmetric curve, and coplanar (collinear) knots a plane curve (straight line). The approach to collinearity/coplanarity must be stable, and the curve must "look reasonable", or fair.

All these requirements are met by a device where the two free vectors associated with each knot are replaced by four independent "guide vectors". Guide vectors at $\vec{V}_i$ depend only on $\vec{V}_{i+1}$ and $\vec{V}_{i-1}$, and parametrise the two free vectors linearly. As will become clear, the reasons for replacing two vectors by four include ease of interpretation, and symmetry.

For specification of guide vectors at each knot we suggest a one-parameter form, where the free parameter ξ determines essentially local scalar curvature κ. This has the advantage of improving control of detail and allows an aesthetic degree of freedom.

Given the knots, plus a choice of the ξ parameter for each, a unique open curve is specified if two "guard knots" are added, one at each end, to fix the guide vectors at the two proper end knots. A smoothly closed curve is a special case when the last three knots and the first three coincide. Some examples are included.

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2. CONTINUITY CONDITIONS

For a curve $\vec{r}(\alpha)$ (α = scalar parameter) we define velocity $\vec{v} \equiv \frac{d\vec{r}}{d\alpha}$ and acceleration $\vec{a} \equiv \frac{d\vec{v}}{d\alpha}$. Then the curve has tangent [2]

$$\vec{t} = \vec{v}/|\vec{v}| \tag{2.1}$$

and curvature

$$\kappa\vec{n} = \vec{v}\wedge\vec{a}\wedge\vec{v}/(\vec{v}.\vec{v})^2. \tag{2.2}$$

The join between two curve segments (say $\vec{r}_1$ and $\vec{r}_2$) will be obvious unless $\vec{t}_1 = \vec{t}_2$. This is clearly achieved by

$$\vec{v}_2 = \lambda\vec{v}_1 \tag{2.3}$$

with the sign of λ chosen to eliminate a cusp.
For complete visual smoothness we demand also that $\kappa_1\vec{n}_1 = \kappa_2\vec{n}_2$. This is obeyed provided

$$\vec{a}_2 = \lambda^2\vec{a}_1 + \mu\vec{v}_1 \tag{2.4}$$

where λ is the same as in (2.3) and μ is arbitrary. Eq. (2.4) is easy to prove by expanding $\vec{a}_2$ on the basis provided by $\vec{v}_1$, $\vec{a}_1$ and $\vec{v}_1\wedge\vec{a}_1$, and using (2.2) and (2.3). It was apparently first published by Manning [3].

3. CURVE SEGMENTS

Curve segments are parametric quintic polynomial forms

$$\vec{r}_i(\alpha) = \vec{V}_{i-1}\beta^5 + 5\vec{B}_{i-1}\beta^4\alpha + 10\vec{D}_{i-1}\beta^3\alpha^2 + 10\vec{C}_i\beta^2\alpha^3 + 5\vec{A}_i\beta\alpha^4 + \vec{V}_i\alpha^5, \tag{3.1}$$

where $\alpha + \beta = 1$ and $\alpha \in [0,1]$. This curve joins the points $\vec{r}_i(0) = \vec{V}_{i-1}$ and $\vec{r}_i(1) = \vec{V}_i$, with its shape controlled by the coefficient vectors $\vec{B}_{i-1}$, $\vec{D}_{i-1}$, $\vec{C}_i$ and $\vec{A}_i$
The tangent to the curve is determined by the vectors joining the points defined by successive coefficients in (3.1), because

$$\frac{1}{5}\vec{v}_i(\alpha) = (\vec{B}_{i-1} - \vec{V}_{i-1})\beta^4 + 4(\vec{D}_{i-1} - \vec{B}_{i-1})\beta^3\alpha + 6(\vec{C}_i - \vec{D}_{i-1})\beta^2\alpha^2 + 4(\vec{A}_i - \vec{C}_i)\beta\alpha^3 + (\vec{V}_i - \vec{A}_i)\alpha^4 \tag{3.2}$$

The scalar curvature is

$$\kappa = |\vec{v}\wedge\vec{a}|/(\vec{v}.\vec{v})^{3/2}, \tag{3.3}$$

where we have

$$\frac{1}{20}\vec{a}_i(\alpha) = (\vec{D}_{i-1} - 2\vec{B}_{i-1} + \vec{V}_{i-1})\beta^3 + 3(\vec{C}_i - 2\vec{D}_{i-1} + \vec{B}_{i-1})\beta^2\alpha + 3(\vec{A}_i - 2\vec{C}_i + \vec{D}_{i-1})\beta\alpha^2 + (\vec{V}_i - 2\vec{A}_i + \vec{C}_i)\alpha^3 \tag{3.4}$$

Therefore κ is zero if the points $\vec{V}_{i-1}$, $\vec{B}_{i-1}$, $\vec{D}_{i-1}$, $\vec{C}_i$, $\vec{A}_i$ and $\vec{V}_i$ are collinear, when (3.1) is just a straight line joining $\vec{V}_{i-1}$ to $\vec{V}_i$. If also the intermediate points $\vec{B}_{i-1}$, $\vec{D}_{i-1}$, $\vec{C}_i$ and $\vec{A}_i$ are equally spaced in their natural order then acceleration $\vec{a}$ is zero, velocity $\vec{v}$ is constant, and (3.1) is linear in α.
The torsion is

$$\tau = \frac{d\vec{a}}{d\alpha}\cdot(\vec{v}\wedge\vec{a})/(\vec{v}\wedge\vec{a})\cdot(\vec{v}\wedge\vec{a}),$$

which is zero if $\vec{V}_{i-1}$, $\vec{B}_{i-1}$, $\vec{D}_{i-1}$, $\vec{C}_i$, $\vec{A}_i$ and $\vec{V}_i$ are coplanar, when therefore the curve (3.1) lies in the plane so defined.
The quantities κ and τ measure the bending and twisting of the curve and the degree of such distortion is indicated by the departure of the coefficients from collinearity/coplanarity. We see that the size of κ can be judged by the relative areas of the triangles having three vector coefficients as vertices, while similarly the magnitude of τ depends on the volumes of tetrahedra erected on four such points. The sketch in fig. 1 is intended to illustrate the typical relationship between a curve segment (3.1) and its defining vectors.

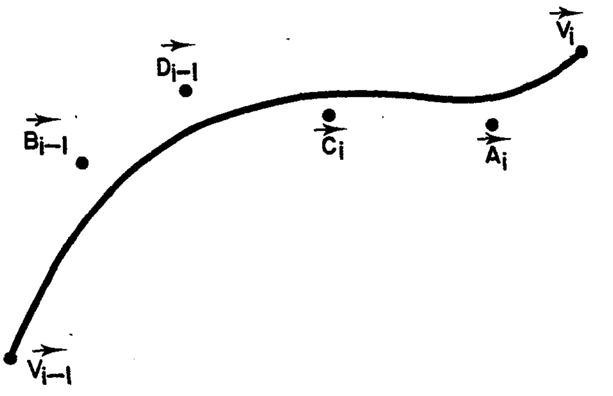

Fig. 1. Sketch of a curve segment (3.1). Each vector coefficient defines a point in space as indicated.

4. JOINING CURVE SEGMENTS

For a curve segment $\vec{r}_i(\alpha)$ of the form (3.1) we have from (3.2)

$$\frac{1}{5}v_i(0) = \vec{B}_{i-1} - \vec{V}_{i-1}$$
$$\frac{1}{5}v_i(1) = \vec{V}_i - \vec{A}_i \tag{4.1}$$

and from (3.4)

$$\frac{1}{20}\vec{a}_i(0) = \vec{D}_{i-1} - 2\vec{B}_{i-1} + \vec{V}_{i-1}$$
$$\frac{1}{20}\vec{a}_i(1) = \vec{V}_i - 2\vec{A}_i + \vec{C}_i. \tag{4.2}$$

Then for two such segments $\vec{r}_i$ and $\vec{r}_{i+1}$ to join smoothly at their common knot

$$\vec{V}_i = \vec{r}_i(1) = \vec{r}_{i+1}(0) \tag{4.3}$$

we must have for tangent continuity from (2.3)

$$\vec{B}_i - \vec{V}_i = \lambda_i(\vec{V}_i - \vec{A}_i) \tag{4.4}$$

and for curvature continuity from (2.4)

$$\vec{D}_i - 2\vec{B}_i + \vec{V}_i = \lambda_i^2(\vec{V}_i - 2\vec{A}_i + \vec{C}_i) + \mu_i(\vec{V}_i - \vec{A}_i), \tag{4.5}$$

where $\lambda_i \geqslant 0$. See fig. 2.

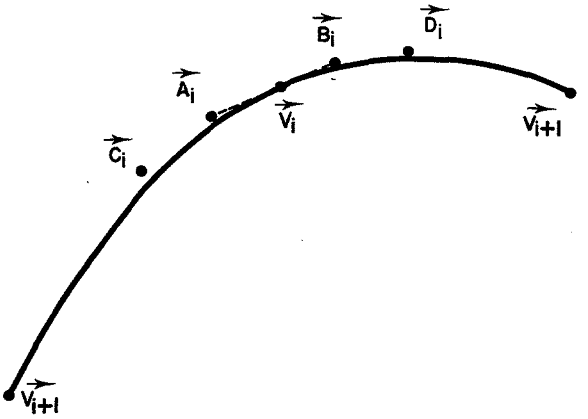

Fig. 2. Curve segments $\vec{r}_i$ and $\vec{r}_{i+1}$ joining smoothly at their common knot $\vec{V}_i$. Points $\vec{A}_i$, $\vec{V}_i$ and $\vec{B}_i$ are collinear for tangent continuity.

We note that the imposition of eqs. (4.4) and (4.5) at each end of a given segment (3.1) involves disjoint sets of the coefficient vectors. Thus a locally defined curve is possible.

Given $\vec{V}_i$ and the two scalars λ_i and μ_i, equations (4.3) to (4.5) leave free at each knot two vectors which can be either $\vec{A}_i$ or $\vec{B}_i$, plus either $\vec{C}_i$ or $\vec{D}_i$. There is an awkward lack of manifest symmetry, in view of the natural desire that a set of knots should specify a unique curve, no matter whether their order is completely reversed. We must parametrise the constrained vectors $\vec{A}_i$, $\vec{B}_i$, $\vec{C}_i$ and $\vec{D}_i$ in a way that restores desirable symmetry, and leads to fair and locally defined curves.

5. PARAMETRISATION BY GUIDE VECTORS

To parametrise the join between segments $\vec{r}_i$ and $\vec{r}_{i+1}$ we propose the following device. Introduce four "guide vectors" $\vec{P}_i$, $\vec{Q}_i$, $\vec{R}_i$ and $\vec{S}_i$, and for some fixed choice of them and the two scalars λ_i and μ_i minimise

$$\&_i \equiv |\vec{A}_i - \vec{P}_i|^2 + |\vec{B}_i - \vec{Q}_i|^2 + |\vec{C}_i - \vec{R}_i|^2 + |\vec{D}_i - \vec{S}_i|^2 \tag{5.1}$$

with respect to $\vec{A}_i$ and $\vec{C}_i$, subject to $\vec{B}_i$ and $\vec{D}_i$ being determined from $\vec{A}_i$ and $\vec{C}_i$ through eqs. (4.4) and (4.5).

This has a useful interpretation which allows fair curves to be produced with a straightforward choice of guide vectors. That is, we visualise each of the pairs of points $(\vec{A}_i, \vec{P}_i)$, $(\vec{B}_i, \vec{Q}_i)$ ... etc. connected by an ideal elastic string, with equations (4.4) and (4.5) representing certain points in fig. 2 joined by ideal rigid rods. For example, (4.4) says that $\vec{V}_i$ is a free pivot for the rod $\vec{A}_i\vec{B}_i$, dividing it in the ratio $1 : \lambda_i$ while $\vec{A}_i$ is tethered to $\vec{P}_i$, and $\vec{B}_i$ to $\vec{Q}_i$. Likewise, (4.5) further imposes a slightly more complicated condition fixing the rods $\vec{V}_i\vec{D}_i$, $\vec{V}_i\vec{A}_i$ and $\vec{V}_i\vec{C}_i$ relative to each other. Then the quantity $\&_i$ defined in eq. (5.1) is minimised when the system is in equilibrium.

A simple calculation gives the unique solution as follows, where the common subscript i is dropped for clarity :

$$\begin{aligned} \vec{A} &= \frac{1}{\Delta}\{(1+\lambda^4)\vec{X} - \frac{\sigma}{\lambda^2}\vec{Y}\}, \\ \vec{C} &= \frac{1}{\Delta}\{\sigma\lambda^2\vec{X} + (1+\lambda^2)\vec{Y}\}, \end{aligned} \tag{5.2}$$

where

$$\Delta = (1+\lambda^4)(1+\lambda^2) + \sigma^2,$$

$$\begin{aligned} \vec{X} &= \vec{P} - \lambda\vec{Q} + \frac{\sigma}{\lambda^2}\vec{R} + \lambda(1+\lambda)\vec{V}, \\ \vec{Y} &= \vec{R} + \lambda^2\vec{S} - \lambda^2(\sigma + 1 - \lambda^2)\vec{V}, \end{aligned} \tag{5.3}$$

and $\sigma = 2\lambda(1+\lambda) + \mu$. Also from (4.4) and (4.5) we have

$$\vec{B} = (1+\lambda)\vec{V} - \lambda\vec{A}$$

$$D = (\sigma + 1 - \lambda^2)\vec{V} - \sigma\vec{A} + \lambda^2\vec{C}.$$

Here only $\vec{X}_i$ and $\vec{Y}_i$ are independent of course, but parametrisation by $\vec{P}_i$, $\vec{Q}_i$, $\vec{R}_i$ and $\vec{S}_i$ leads to restoration of symmetry, maintenance of locality, and a framework for producing 'fair' curves.

6. CHOICE OF PARAMETERS

At this stage to specify the smooth curve through a given sequence of knots it is necessary to fix at each $\vec{V}_i$ the scalars λ_i and μ_i, and the guide vectors $\vec{P}_i$, $\vec{Q}_i$, $\vec{R}_i$ and $\vec{S}_i$.

We set all $\mu_i \equiv 0$, and assuming immediately successive knots distinct, choose

$$\lambda_i = |\vec{V}_{i+1} - \vec{V}_i| \,/\, |\vec{V}_i - \vec{V}_{i-1}|. \tag{6.1}$$

This ensures symmetry under complete reversal of the sequence of knots, and allows proportional spacing of coefficient points $\vec{A}_i$, $\vec{B}_i$, $\vec{C}_i$, $\vec{D}_i$ according to the spacing

of knots.
Because eq. (6.1) links only three adjacent knots the locality of the construction is preserved. Changing $\vec{V}_i$ affects only $\vec{A}_i$, $\vec{B}_i$, $\vec{C}_i$ and $\vec{D}_i$, and λ_i and $\lambda_{i\pm1}$. Thus only four neighbouring segments are affected, two on each side. The two further segments are the less affected, through λ-factors only.
To maintain maximal locality the guide vectors at $\vec{V}_i$ should depend, like λ_i, only on $\vec{V}_i$ and $\vec{V}_{i\pm1}$. Moreover they must be chosen to produce a fair curve, and this is where the rigid rod/elastic string model helps. When $\vec{V}_{i-1}$, $\vec{V}_i$ and $\vec{V}_{i+1}$ are almost collinear, we demand that in the vicinity of $\vec{V}_i$ the curve itself is correspondingly close to a straight line. From Section 3, this is ensured if $\vec{C}_i$ and $\vec{A}_i$ are close to points respectively 3/5 and 4/5 the way along a straight line joining $\vec{V}_{i-1}$ to $\vec{V}_i$, while $\vec{B}_i$ and $\vec{D}_i$ are likewise near to points 1/5 and 2/5 the way from $\vec{V}_i$ to $\vec{V}_{i+1}$ on a straight line, as eq. (6.1) provides.
Thus the model of rigid rods and elastic strings suggests

$$\vec{P}_i, \vec{Q}_i = \frac{1}{5}(4\vec{V}_i + \vec{V}_{i\mp1}),$$
$$\vec{R}_i, \vec{S}_i = \frac{1}{5}(3\vec{V}_i + 2\vec{V}_{i\mp1}). \tag{6.2}$$

Equations (6.2) give the desired limit as well as curvature in the correct sense away from it.
However, eqs. (6.2) are acceptable only when the three knots are exactly collinear; in other orientations the result is the sort depicted in fig. 3, with unrealistic variation of curvature. Instead we seek a more satisfactory prescription with eqs. (6.2) as its limit.

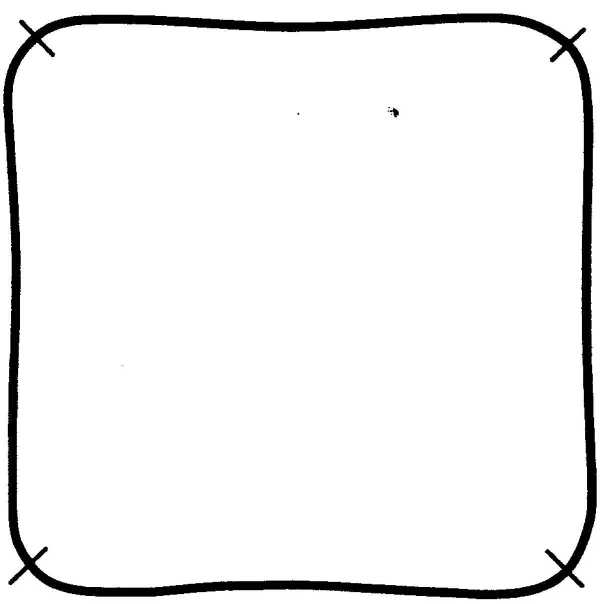

Fig. 3. Closed curve defined by knots at the four corners of a square, with guide vectors chosen by eq. (6.2) - i.e. values $\xi_i = 0.4$ in eq. (6.3).

A simple choice is

$$\vec{P}_i, \vec{Q}_i = \frac{1}{2}\xi_i\vec{U}_i + (1-\frac{1}{2}\xi_i)\vec{V}_i \mp \frac{1}{5}\,\frac{(\lambda_i)}{1+\lambda_i}(\vec{V}_{i-1} - \vec{V}_{i+1}), \tag{6.3}$$
$$\vec{R}_i, \vec{S}_i = \xi_i\vec{U}_i + (1-\xi_i)\vec{V}_i \mp \frac{2}{5}\,\frac{(\lambda_i)}{1+\lambda_i}(\vec{V}_{i-1} - \vec{V}_{i+1}),$$

where $\vec{U}_i = (\lambda_i\vec{V}_{i-1} + \vec{V}_{i+1})/(1+\lambda_i)$ and the bracketed numerator λ-factor goes with the negative sign, ($\vec{P}_i$, $\vec{R}_i$). Fig. 4 explains the logic behind these equations.

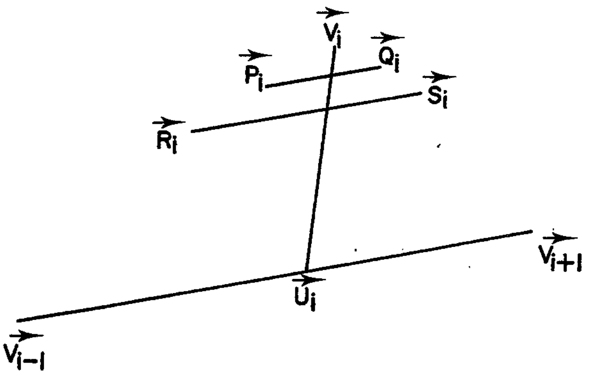

Fig. 4. Illustration of (6.3). The point $\vec{U}_i$ divides $\vec{V}_{i+1} - \vec{V}_{i-1}$ in the ratio $1:\lambda_i$. The line $\vec{S}_i - \vec{R}_i$ is parallel to $\vec{V}_{i+1} - \vec{V}_{i-1}$ and 2/5 its length, dividing $\vec{V}_i - \vec{U}_i$ in the ratio $\xi_i : 1-\xi_i$ and is itself divided $1:\lambda_i$. The line $\vec{Q}_i - \vec{P}_i$ is parallel, half as long, divides in the ratio $1/2\,\xi_i : 1 - 1/2\,\xi_i$ and is also divided $1:\lambda_i$.

The value of the parameter ξ_i controls the curvature at the knot $\vec{V}_i$. If $\xi_i = 2/5$ then eqs. (6.3) reduce to (6.2), and κ at $\vec{V}_i$ is large. Fairer curves result if

$$0 < \xi_i < 2/5 \tag{6.4}$$

as fig. 5 illustrates. If $\xi_i = 0$ the knot and its guide vectors are collinear and $\&_i = 0$. Then the curvature κ at $\vec{V}_i$ is zero; increasing ξ_i increases κ.
As locality implies, $\vec{t}$ at $\vec{V}_i$ is parallel to $\vec{V}_{i+1} - \vec{V}_{i-1}$, and $\vec{n}$ lies in the plane of the three knots, pointing in the correct sense if (6.4) is obeyed. The coplanarity of $\vec{V}_i$, $\vec{V}_{i\pm1}$, and $\vec{P}_i$, $\vec{Q}_i$, $\vec{R}_i$ and $\vec{S}_i$ does not imply any special constraint on the torsion τ at $\vec{V}_i$.
At this stage we have a complete specification for constructing a smooth curve through a given set of points. At each point or knot $\vec{V}_i$ a scalar curvature parameter ξ_i is needed, and for an open curve we must add a guard knot at each end, chosen to determine a suitable set of guide vectors for each proper end knot. A smoothly closed curve needs the first guard knot to be the same as the second-to-last proper knot, and the last guard knot to coincide with the second proper knot. The construction is direct and extremely fast, and production

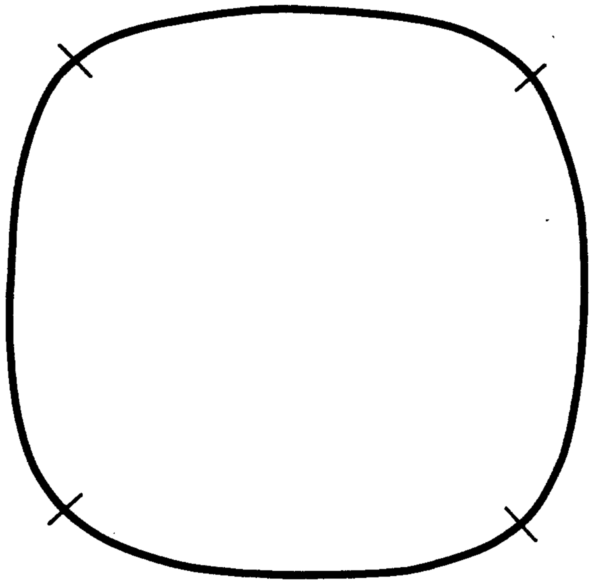

Fig. 5. As fig. 3, but with $\xi_i = 0.1$ at all knots.

of a given segment requires information about only four knots; hence computer storage demands are very small.

7. APPLICATIONS

The test of a scheme for summarising curves by sets of representative points is to see how well it reproduces given shapes. We give three examples.
Firstly, fig. 6 shows an open plane curve represented by eight knots $\vec{V}_0$ to $\vec{V}_7$, plus two guard knots. The hill and valley outline was drawn freehand, just to

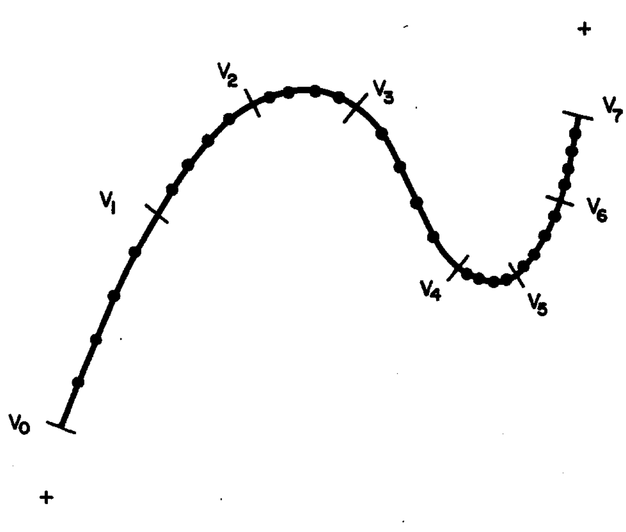

Fig. 6. The continuous line is a freely drawn curve. Knots V_0 to V_7, and the 2 guard knots, chosen by eye, are shown. These points, plus the values $\xi_i = 0.1$, give the interpolation indicated by dots. Each dot is at an interval of 0.2 in the curve parameter α, where a segment $\vec{r}_i(\alpha)$ has $\alpha \in [0,1]$.

show some changes of curvature including an inflection. Knots were placed by eye - one at each end, one near each turning point, and intermediate (and guard) knots were distributed bearing in mind that $\vec{V}_{i+1} - \vec{V}_{i-1}$ controls the tangent at $\vec{V}_i$. The knot coordinates, read from superimposed transparent graph paper, plus values of parameters $\xi_i = 0.1$ ($i = 0$ to 7) produce (via FORTRAN program) a smooth interpolation whose agreement with the original curve is indicated.
The faithfulness of the reproduction is excellent. The only significant discrepancies are in segments $\vec{r}_3$ and $\vec{r}_4$, at the top of the peak and near the point of inflection. By adjusting $\vec{V}_3$ and ξ_3 it is easy to make the necessary local corrections and get a visually perfect fit. The use of a computer-controlled CRT display is ideal here.
The great advantage of locality is evident at this stage of the construction. The changes in segments $\vec{r}_3$ and $\vec{r}_4$ are accompanied by only minor perturbations in $\vec{r}_2$ and $\vec{r}_5$, and the already acceptable description of the curve in segments $\vec{r}_1$, $\vec{r}_6$ and $\vec{r}_7$ is unaffected. Moreover the influences of $\vec{V}_i$ and ξ_i on local tangent and curvature are always clear. We note the usefulness of the freedom to make minor adjustments by altering the parameters ξ_i, but emphasise that clearly the present extreme locality means that care is needed near points of inflection.
We give two further examples, dealing with genuine space curves, and testing the construction by choosing the knots less carefully.
The second example is the regular helix

$$\vec{r}(\alpha) = (\alpha, \cos 2\pi\alpha, \sin 2\pi\alpha)$$

represented by regularly-spaced knots at $\alpha = 0, 1/4, 1/2, 3/4, \ldots$ and by all curvature parameters $\xi_i = 0.1$. Even with such arbitrary values, the reproduction is good. By comparing y with $\cos 2\pi x$ and z with $\sin 2\pi x$ for points (x, y, z) on the curve, the deviation from a helix is less than 1 %.
Thirdly, the twisted cubic

$$\vec{r}(\alpha) = (\alpha, \alpha^2, \alpha^3)$$

represented by knots at $\alpha = 0, 1, 2, 3, \ldots$ and all $\xi_i = 0.1$ is reproduced to three significant figures for $\alpha > 3$, and to four significant figures for $\alpha > 5$. The arbitrarily-chosen knots mean that there are discrepancies in the third significant figure for $1 < \alpha < 2$ where curvature and torsion vary rapidly. (Comparison here is between x^2 and y, and x^3 and z.)

8. CONCLUSION

Summarising, we propose a scheme for space-curve interpolation which is simple and direct, requiring relatively little computer time and storage. It produces fair and reasonable curves with manifest Euclidean and scale invariance, and with the important property of locality. Locality implies convenient control of detail, helped by the further freedom (in this version) to choose a curvature parameter at each knot. An interesting extension of the present ideas to the patching of surfaces appears possible.

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